

Dyadic A_1 weights and equimeasurable rearrangement of functions

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Abstract: We prove that the decreasing rearrangement of a dyadic A_1 -weight w with dyadic A_1 constant $[w]_1^{\mathcal{T}} = c$ with respect to a tree \mathcal{T} of homogeneity k , on a non-atomic probability space, is a usual A_1 weight on $(0, 1]$ with A_1 -constant $[w^*]_1$ not more than $kc - k + 1$. We prove also that the result is sharp, when one considers all such weights w .

Keywords: Dyadic, weight, rearrangement.

1. Introduction

The theory of Muckenhoupt weights has been proved to be an important tool in analysis due to their self-improving properties (see [2, 3, 8]).

One class of special interest is $A_1(J, c)$ where J is an interval on \mathbb{R} and c a constant $c \geq 1$. Then $A_1(J, c)$ is defined as the class of all non-negative locally integrable functions w defined on J , such that for every subinterval $I \subseteq J$ we have that

$$\frac{1}{|I|} \int_I w(y) dy \leq c \operatorname{ess\,inf}_{x \in I} w(x) \quad (1.1)$$

where $|\cdot|$ is the Lebesgue measure on \mathbb{R} .

In [1] it is proved that if $w \in A_1(J, c)$ then $w^* \in A_1((0, |J|], c)$, where w^* is the non-increasing rearrangement of w . That is $w \in A_1(J, c)$ gives that

$$\frac{1}{t} \int_0^t w^*(y) dy \leq c w^*(t), \quad (1.2)$$

for every $t \in (0, |J|]$.

Here for a $w : J \rightarrow \mathbb{R}^+$, w^* stands for

$$w^*(t) = \sup_{\substack{e \subseteq J \\ |e| \geq t}} \inf_{x \in e} w(x), \quad \text{for any } t \in (0, |J|].$$

The fact mentioned above helps (as one can see in [1]) in the determination of all p such that $p > 1$ and $w \in RH_p^J(c')$ for some $1 \leq c' < +\infty$ whenever $w \in A_1(J, c)$, where by $RH_p^J(c')$ we mean the class of all weights w defined on J which satisfy a reverse Holder inequality with constant c' upon all the subintervals $I \subseteq J$. One can also see related problems for estimates for the range of p in higher dimensions in [4] and [5].

In this paper we are interested for the opposite dyadic case. A way of studying dyadic A_1 weights is by using the respective dyadic maximal operator.

More precisely, a locally integrable non-negative function w on \mathbb{R}^n is called a dyadic A_1 weight if it satisfies the following condition

$$\frac{1}{|Q|} \int_Q w(y) dy \leq c \operatorname{ess\,inf}_{x \in I} w(x), \quad (1.3)$$

for every dyadic cube on \mathbb{R}^n .

This condition is equivalent to the inequality

$$\mathcal{M}_d w(x) \leq c w(x), \quad (1.4)$$

for almost all $x \in \mathbb{R}^n$. Here \mathcal{M}_d is the dyadic maximal operator defined by

$$\mathcal{M}_d w(x) = \sup \left\{ \frac{1}{|Q|} \int_Q w(y) dy : x \in Q, Q \subset \mathbb{R}^n \text{ is a dyadic cube} \right\}. \quad (1.5)$$

The smallest $c \geq 1$ for which (1.3) (equivalently (1.4)) holds is called the dyadic A_1 constant of w and is denoted by $[w]_1^d$.

Let us now fix a dyadic cube Q on \mathbb{R}^n . A natural problem that arises is the behaviour of $(w/Q)^* : (0, |Q|] \rightarrow \mathbb{R}^+$ when one knows that $[w]_1^d = c$. It turns out that $(w/Q)^*$ is a usual A_1 weight on $(0, |Q|]$ with constant not more than $2^n c - 2^n + 1$.

More precisely we will prove the following

Theorem 1. *Let w be a dyadic A_1 weight on \mathbb{R}^n with dyadic A_1 constant $[w]_1^d = c$. Let Q be a fixed dyadic cube on \mathbb{R}^n . Then the following inequality is satisfied*

$$\frac{1}{t} \int_0^t (w/Q)^*(y) dy \leq (2^n c - 2^n + 1) (w/Q)^*(t), \quad (1.6)$$

for every $t \in (0, |Q|]$.

Moreover the last inequality is sharp when one considers all dyadic A_1 weights with $[w]_1^d = c$. ■

We remark that by using a standard dilation argument it suffices to prove (1.6) for $Q = [0, 1]^n$ and for all functions w defined only on $[0, 1]^n$ and satisfying the A_1

condition only for dyadic cubes contained in $[0, 1]^n$. Actually, we will work on more general non-atomic probability spaces (X, μ) equipped with a structure \mathcal{T} similar to the dyadic one. (We give the precise definition in the next section).

The paper is organized as follows:

In Section 2. we give some tools needed for the proof of Theorem 1. These are obtained from [6] and [7].

In Section 3 we give the proof of Theorem 1 in it's general form (as Theorem 2) and mention two applications of it.

2. Preliminaries

We fix a non-atomic probability space (X, μ) and a positive integer $k \geq 2$.

We give the following

Definition 1. *A set of measurable subsets of X will be called a tree of homogeneity k if*

- i) For every $I \in \mathcal{T}$ there corresponds a subset $C(I) \subseteq \mathcal{T}$ containing exactly k pairwise disjoint subsets of I such that $I = \cup C(I)$ and each element of $C(I)$ has measure $(1/k)\mu(I)$.*
- ii) $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_{(m)}$ where $\mathcal{T}_{(0)} = \{X\}$ and $\mathcal{T}_{(m+1)} = \bigcup_{I \in \mathcal{T}_{(m)}} C(I)$.*
- iii) The tree \mathcal{T} differentiates $L^1(X, \mu)$, that is if $\varphi \in L^1(X, \mu)$ then for μ almost all $x \in X$ and every sequence $(I_k)_{k \in \mathbb{N}}$ such that $x \in I_k$, $I_k \in \mathcal{T}$ and $\mu(I_k) \rightarrow 0$ we have that*

$$\varphi(x) = \lim_{k \rightarrow +\infty} \frac{1}{\mu(I_k)} \int_{I_k} \varphi d\mu. \quad \blacksquare$$

It is clear that each family $\mathcal{T}_{(m)}$ consists of k^m pairwise disjoint sets, each having measure k^{-m} , whose union is X .

Moreover, if $I, J \in \mathcal{T}$ and $I \cap J$ is non empty then $I \subseteq J$ or $J \subseteq I$.

For this family \mathcal{T} we define the associated maximal operator $\mathcal{M}_{\mathcal{T}}$ by

$$\mathcal{M}_{\mathcal{T}}\varphi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\varphi| d\mu : x \in I \in \mathcal{T} \right\}, \quad (2.1)$$

for any $\varphi \in L^1(X, \mu)$ and we will say that a non-negative integrable function w is an A_1 weight with respect to \mathcal{T} if

$$\mathcal{M}_{\mathcal{T}}\varphi(x) \leq C\varphi(x), \quad (2.2)$$

for almost every $x \in X$. The smallest constant C for which (2.2) holds will be called the A_1 constant of w with respect to \mathcal{T} and will be denoted by $[w]_1^\mathcal{T}$.

We give now the following:

Definition 2. Every non-constant function w of the form $w = \sum_{P \in \mathcal{T}_{(m)}} \lambda_P \xi_P$, for a specific $m > 0$, and for positive λ_P , will be called a \mathcal{T} -step function. (ξ_P denotes the characteristic function of P). ■

It is then clear that every \mathcal{T} -step function is an A_1 weight with respect to \mathcal{T} . Let $\delta = 1/[w]_1^\mathcal{T}$, $0 < \delta < 1$ and for any $I \in \mathcal{T}$ write $Av_I(w) = \frac{1}{\mu(I)} \int_I w d\mu$.

Now for every $x \in X$, let $I_w(x)$ denote the largest element of the set $\{I \in \mathcal{T} : x \in I \text{ and } \mathcal{M}_\mathcal{T} w(x) = Av_I(w)\}$ (which is non-empty since $Av_J(w) = Av_P(w)$ for every $P \in \mathcal{T}_{(m)}$ and $J \subseteq P$).

Next for any $I \in \mathcal{T}$ we define the set

$$A_I = A(w, I) = \{x \in X : I_w(x) = I\}$$

and let $S = S_w$ denote the set of all $I \in \mathcal{T}$ such that A_I is non-empty. It is clear that each such A_I is a union of certain P from $\mathcal{T}_{(m)}$ and moreover

$$\mathcal{M}_\mathcal{T} w = \sum_{I \in S} Av_I(w) \xi_{A_I}.$$

We also define the correspondence $I \rightarrow I^*$ with respect to S as follows: I^* is the smallest element of $\{J \in S_w : I \subsetneq J\}$. This is defined for every $I \in S$ that is not maximal with respect to \subseteq .

We recall parts of two Lemmas from [6].

Lemma 1. For all $I \in \mathcal{T}$ we have $I \in S$, if and only if, $Av_Q(w) < Av_I(w)$ whenever $I \subseteq Q \in \mathcal{T}$, $I \neq Q$. In particular $X \in S$ and so $I \rightarrow I^*$ is defined for all $I \in S$ such that $I \neq X$. ■

Lemma 2. Let $I \in S$. Then, if $J \in S$ is such that

$$J^* = I \text{ then } y_I < y_J \leq (k - (k - 1)\delta)y_I. \quad \blacksquare$$

3. Main theorem and proof

In this section we will prove the following.

Theorem 2. Let \mathcal{T} be a tree of homogeneity $k \geq 2$ on the probability non-atomic space (X, μ) , and let w be A_1 weight with respect to \mathcal{T} with A_1 -constant $[w]_1^{\mathcal{T}} = c$. Then if one considers $w^* : (0, 1] \rightarrow \mathbb{R}^+$ we have that $\frac{1}{t} \int_0^t w^*(y) dy \leq (kc - k + 1)w^*(t)$, for every $t \in (0, 1]$, where as usual w^* is defined by $w^*(t) = \sup_{\substack{e \subseteq X \\ \mu(e) \geq t}} \inf_{x \in e} w(x)$, $t \in (0, 1]$.

Moreover the constant appearing in the right of the last inequality is sharp, if one considers all A_1 weights with respect to \mathcal{T} with constant $[w]_1^{\mathcal{T}} = c$. ■

Proof. We suppose for the beginning that w is a \mathcal{T} -step function. Fix $t \in (0, 1]$ and consider the set

$$\begin{aligned} E_t &= \{x \in X : \mathcal{M}_{\mathcal{T}}w(x) > cw^*(t)\} \\ &= \{\mathcal{M}_{\mathcal{T}}w > c\lambda\}, \text{ where } \lambda = w^*(t). \end{aligned}$$

Then E_t is a measurable subset of X . We first assume that $\mu(E_t) > 0$.

We consider the family of all those $I \in \mathcal{T}$ maximal under the condition $Av_I(w) > c\lambda$, and denote it by $(I_j)_j$. Then $(I_j)_j$ is pairwise disjoint and $E_t = \cup I_j$.

Additionally for every j and $I \in \mathcal{T}$ such that $I \supsetneq I_j$ we have that $\frac{1}{\mu(I)} \int_I w d\mu = Av_I(w) \leq c\lambda$ because of the maximality of I_j .

In view of Lemma 1 this gives $I_j \in S_w = S$, for every j .

For every I_j consider $I_j^* \in S$. Then by Lemma 2, $y_{I_j} \leq [k - (k - 1)\delta]y_{I_j^*}$, where $\delta = 1/c$ and of course $y_{I_j^*} \leq c\lambda$. So, we have that

$$y_{I_j} \leq [k - (k - 1)\delta]c\lambda = (kc - k + 1)\lambda, \text{ for every } j.$$

This gives

$$\begin{aligned} \int_{I_j} w d\mu &\leq (kc - k + 1)\lambda\mu(I_j) \Rightarrow \int_{E_t} w d\mu \leq (kc - k + 1)\lambda\mu(E_t) \\ &\Rightarrow \frac{1}{\mu(E_t)} \int_{E_t} w d\mu \leq (kc - k + 1)\lambda. \end{aligned} \quad (3.1)$$

Since $\mathcal{M}_{\mathcal{T}}w \leq cw$ on X , and $E_t = \{\mathcal{M}_{\mathcal{T}}w > c\lambda\}$ we obviously have $E_t \subseteq \{w > \lambda\} = \{w > w^*(t)\}$.

There exist now $E_t^* \subseteq (0, 1]$ Lebesgue measurable such that $|E_t^*| = \mu(E_t) =: t_1$, and such that $\int_{E_t^*} w^*(y) dy = \int_{E_t} w d\mu$. Obviously we can arrange everything in a way such that $E_t^* \subseteq \{w^* > w^*(t)\} \subseteq (0, t)$. As a result $t_1 \leq t$.

Since now \mathcal{T} differentiates $L^1(X, \mu)$ we have that almost every element of the set $\{w > c\lambda\} \subseteq X$ belongs to E_t .

Since $\mu(E_t) > 0$ we have that $\mu(\{w > c\lambda\}) > 0$.

Let now t_2 be such that

$$w^*(t) > \lambda c \text{ for every } t \in (0, t_2) \text{ and } w^*(t) \leq c\lambda, \text{ for every } t \in (t_2, 1).$$

Then, we can arrange everything (by deleting suitable sets of Lebesgue measure zero) in a way that $E_t^* = (0, t_2) \cup A_t$, where A_t is a Lebesgue measurable subset of (t_2, t) and $|A_t| = t_1 - t_2$ (Of course $t_2 = |(0, t_2)| = |\{w^* > \lambda c\}| = \mu(\{w > \lambda c\}) \leq \mu(\{\mathcal{M}_{\mathcal{T}} w > \lambda c\}) = \mu(E_t) =: t_1$).

We will prove the following

$$\frac{1}{\mu(E_t)} \int_{E_t} w d\mu \geq \frac{1}{t} \int_0^t w^*(y) dy, \quad (3.2)$$

(3.2) is equivalent to

$$\begin{aligned} \frac{1}{t_1} \int_{E_t^*} w^*(y) dy &\geq \frac{1}{t} \int_0^t w^*(y) dy \Leftrightarrow t \int_0^{t_2} w^*(y) dy + t \int_{A_t} w^*(y) dy \\ &\geq t_1 \int_0^{t_2} w^*(y) dy + t_1 \int_{t_2}^t w^*(y) dy \\ &\Leftrightarrow (t - t_1) \int_0^{t_2} w^*(y) dy + t \int_{A_t} w^*(y) dy \\ &\geq t_1 \int_{t_2}^t w^*(y) dy, \end{aligned} \quad (3.3)$$

We define $\Gamma_t = (t_2, t) \setminus A_t$. (3.3) then becomes

$$\begin{aligned} (t - t_1) \int_0^{t_2} w^*(y) dy + (t - t_1) \int_{A_t} w^*(y) dy &\geq t_1 \int_{\Gamma_t} w^*(y) dy \\ \Leftrightarrow (t - t_1) \int_{E_t^*} w^*(y) dy &\geq t_1 \int_{\Gamma_t} w^*(y) dy. \end{aligned} \quad (3.4)$$

But of course

$$\int_{E_t^*} w^*(y) dy = \int_{E_t} w d\mu > \mu(E_t) \cdot c\lambda = c\lambda \cdot t_1,$$

in view of the known weak type inequality for $\mathcal{M}_{\mathcal{T}}$, namely:

$$\mu(\{\mathcal{M}_{\mathcal{T}} \varphi > \lambda\}) < \frac{1}{\lambda} \int_{\{\mathcal{M}_{\mathcal{T}} \varphi > \lambda\}} \varphi.$$

So, if we prove that

$$\int_{\Gamma_t} w^*(y) dy \leq c\lambda(t - t_1), \quad (3.5)$$

we complete the proof of (3.2). But (3.5) is obvious since $w^*(y) \leq c\lambda$ on (t_2, t) , $I_t \subseteq (t_2, t)$ and

$$|I_t| = |(t_2, t) \setminus (A_t)| = (t - t_2) - |A_t| = t - t_1.$$

We have thus proved for every w \mathcal{T} -step function and t such that $\mu(\{\mathcal{M}_{\mathcal{T}}w > c \cdot w^*(t)\}) > 0$, that

$$\frac{1}{t} \int_0^t w^*(y) dy \leq (kc - k + 1)w^*(t). \quad (3.6)$$

If t is such that $\mu(\{\mathcal{M}_{\mathcal{T}}w > cw^*(t)\}) = 0$ then obviously $\mathcal{M}_{\mathcal{T}}w(x) \leq cw^*(t)$, for almost all $x \in X$, so since \mathcal{T} differentiates $L^1(X, \mu)$: $w(y) \leq cw^*(t)$ for almost all $y \in X$. This obviously give (3.6) since $c \leq kc - k + 1$.

Additionally if w is in general an A_1 -weight with respect to \mathcal{T} , then an approximation argument by \mathcal{T} -simple A_1 -weights gives the result for w .

More precisely one can easily see, that if w is a A_1 weight with respect to \mathcal{T} , with A_1 -constant $[w]_1^{\mathcal{T}} = c$ then there exist a sequence of \mathcal{T} -simple functions $(w_n)_n$ increasing as n increases, and such that $w_n \leq w$ and $[w]_1^{\mathcal{T}} = c_n \leq c$ with $w_n \rightarrow w$ and $c_n \rightarrow c$ as $n \rightarrow +\infty$.

In order to finish the proof of Theorem 2 we just need to prove the sharpness of the result. We do it right now:

Fix $k \geq 2$. We suppose that we are given a tree \mathcal{T} of homogeneity k , and consider $\mathcal{T}_{(2)}$. Then

$$\mathcal{T}_{(2)} = \{P_1, \dots, P_k, P_{k+1}, \dots, P_{2k}, \dots, P_{k^2-k+1}, \dots, P_{k^2}\} \text{ where}$$

$$\mathcal{T}_{(1)} = \left\{ \bigcup_{i=1}^k P_i, \bigcup_{i=k+1}^{2k} P_i, \dots, \bigcup_{i=k^2-k+1}^{k^2} P_i \right\} = \{I_1, I_2, \dots, I_k\}.$$

We have that $\mu(P_i) = \frac{1}{k^2}$, $\forall i$.

Suppose $\delta > 0$ be such that $\delta < \frac{1}{k^2}$, and consider for any such δ a set A_δ of measure $\mu(A_\delta) = \delta$ such that $A_\delta \subseteq P_1$ ((X, μ) is non atomic). Let $c \geq 1$ and $\alpha, \epsilon < 0$. Let $\varphi = \varphi_\delta$ be the function defined as follows:

$$\begin{aligned} \varphi/A_\delta &:= \alpha \\ \varphi/I_1 \setminus A_\delta &:= \epsilon \\ \varphi/P_{k+1} &:= \alpha, & \varphi/I_2 \setminus P_{k+1} &:= \epsilon \\ \varphi/P_{2k+1} &:= \alpha, & \varphi/I_3 \setminus P_{2k+1} &:= \epsilon \\ &\dots \\ \varphi/P_{k^2-k+1} &:= \alpha, & \varphi/I_k \setminus P_{k^2-k+1} &:= \epsilon \end{aligned}$$

It is easy to see that $\varphi = \varphi_\delta$ is a A_1 weight with A_1 constant

$$c_\delta = [w]_1^\mathcal{T} = \frac{Av_{I_1}(\varphi)}{\epsilon} = \frac{k}{\epsilon} \int_I \varphi d\mu = \frac{k}{\epsilon} \left[a\delta + \left(\frac{1}{k} - \delta \right) \epsilon \right].$$

Then $c_\delta \rightarrow c$, as $\delta \rightarrow 1/k^{2-}$ iff: α, ϵ are chosen such that $kc - k + 1 = \frac{\alpha}{\epsilon}$. (Given k, c). Let us choose α, ϵ be such as mentioned just before, with $\epsilon < \alpha$.

Then $\varphi_\delta^*(1/k) = \epsilon$, so $\varphi_\delta^*(1/k)(kc - k + 1) = \alpha$, while $k \int_0^{1/k} \varphi_\delta^*(y) dy$ tends to α , while $\delta \rightarrow 1/k^{2-}$.

By this we end the proof of Theorem 2. ■

Theorem 1 of Section 1 is an immediate Corollary of Theorem 2.

Additionally the following are consequences of Theorem 2.

Corollary 1. *Let w be an A_1 weight with respect to the tree \mathcal{T} of homogeneity ($k \geq 2$) on (X, μ) with $[w]_1^\mathcal{T} = c$. Then if one considers $((0, 1], |\cdot|)$ equipped with the usual k -adic tree \mathcal{T}_k , where $|\cdot|$ is the Lebesgue measure on $(0, 1]$. Then $[w^*]_1^{\mathcal{T}_k} \leq kc - k + 1$ and this result is sharp.*

Proof. Obvious, according to the function φ_δ constructed at the end of Theorem 2. ■

Corollary 2. *Let w be A_1 -weight on \mathbb{R}^n as described in Section 1. Then $w^* : (0, +\infty) \rightarrow \mathbb{R}^+$ has the following property:*

$$\frac{1}{t} \int_0^t w^*(y) dy \leq (kc - k + 1)w^*(t), \text{ for every } t \in (0, +\infty)$$

and the last inequality is sharp. ■

Proof. We expand \mathbb{R}^n as a union of an increasing sequence $(Q_j)_j$ of dyadic cubes, and use Theorem 2 in any of these. ■

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